

# On the asymptotic stability of steady flows with nonzero flux in two-dimensional exterior domains

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The Navier-Stokes equations in a two-dimensional exterior domain are considered. The asymptotic stability of stationary solutions satisfying a general hypothesis is proven under any  $L^2$ -perturbation. In particular the general hypothesis is valid if the steady solution is the sum of the critically decaying flux carrier with flux  $|\Phi| < 2\pi$  and a small subcritically decaying term. Under the central symmetry assumption, the general hypothesis is also proven for any critically decaying steady solutions under a suitable smallness condition.

**Keywords** Navier-Stokes equations, Stability of steady solutions, Nonzero flux

**MSC class** 35Q30, 35B35, 76D05

## 1 Introduction

We consider the Navier-Stokes equations in a domain  $\Omega = \mathbb{R}^2 \setminus \bar{B}$  where  $B$  is a bounded simply connected Lipschitz domain,

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= \Delta \mathbf{u} - \nabla p, & \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0, & \mathbf{u}|_{\partial\Omega} &= \mathbf{u}^*, \end{aligned} \quad (1)$$

with an inhomogeneous boundary condition  $\mathbf{u}^*$ . Without loss of generality, we will suppose that  $\mathbf{0} \in B$ . Therefore, there exists  $R > \varepsilon > 0$ , such that  $B(\mathbf{0}, \varepsilon) \cap \Omega = \emptyset$  and  $\bar{B} \subset B(\mathbf{0}, R)$ , where  $B(\mathbf{0}, r)$  denotes the open ball of radius  $r$  centered at the origin. Under smallness assumptions on  $\mathbf{u}^*$ , we expect that the long-time behavior of the solution to this system will be close to the corresponding steady-state,

$$\Delta \bar{\mathbf{u}} - \nabla \bar{p} = \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}, \quad \nabla \cdot \bar{\mathbf{u}} = 0, \quad \bar{\mathbf{u}}|_{\partial\Omega} = \mathbf{u}^*. \quad (2)$$

We note that for large values of the forcing  $\mathbf{u}^*$ , the long-time behavior is in general not expected to be close to the steady state due to the presence of turbulence. In all what follows, we will only consider the case where  $\bar{\mathbf{u}}$  decays to zero at infinity,

$$\lim_{|x| \rightarrow \infty} \bar{\mathbf{u}} = \mathbf{0}.$$

The existence of such solutions for this steady-state problem is still open in general [Galdi \(2011, Chapter I\)](#). Some hypotheses on the boundary condition  $\mathbf{u}^*$  ensuring the existence of such a steady solution are explained later on.

For vanishing forcing  $\mathbf{u}^* = \mathbf{0}$  or in  $\mathbb{R}^2$ , [Borchers & Miyakawa \(1992\)](#) established the asymptotic stability of trivial solution  $\bar{\mathbf{u}} = \mathbf{0}$  under  $L^2$ -perturbations. Even under more general hypotheses, the solution is known to be asymptotic to the Oseen vortex ([Gallay & Wayne, 2002, 2005](#); [Iftimie et al., 2011](#); [Gallay & Maekawa, 2013](#); [Ervedoza et al., 2014](#); [Maekawa, 2015](#)). The aim of this paper is to stud the stability of the steady solution for a nonzero forcing  $\mathbf{u}^*$ . More precisely, we introduce a general hypothesis on the steady solutions such that their asymptotic stability with respect to any  $L^2$ -perturbation can be proven by the energy method. We will also determine some explicit criteria on the steady solution implying the validity of the main hypothesis. The proof will follow the ideas developed by [Karch & Pilarczyk \(2011\)](#) for stability of the Landau solutions for the three-dimensional case, the main difference being lemma 16. In particular for  $\bar{\mathbf{u}} = \mathbf{u}^* = \mathbf{0}$ , this later lemma allow us to propose a simpler proof of the results of [Masuda \(1984\)](#); [Borchers & Miyakawa \(1992\)](#).

We now introduce the concept of criticality for the steady solutions ([Guillod, 2015, §1](#)) in order to explain the cases which can be treated by our method. The Navier-Stokes equations (2) in  $\mathbb{R}^2$  have the following scaling symmetry,  $\bar{\mathbf{u}} \mapsto \lambda \bar{\mathbf{u}}(\lambda \mathbf{x})$ . A scale-invariant steady solution  $\bar{\mathbf{u}}$  therefore has to decay like  $|\mathbf{x}|^{-1}$  at infinity, and we call this decay critical. The steady solutions decaying strictly faster than the critical decay, more precisely bounded by  $(|\mathbf{x}| \log |\mathbf{x}|)^{-1}$  is called subcritically decaying. To our knowledge, all the known solutions of (2) are of the form of a critically decaying term plus a subcritically decaying term ([Jeffery, 1915](#); [Hamel, 1917](#); [Galdi, 2004](#); [Yamazaki, 2009, 2011](#); [Pileckas & Russo, 2012](#); [Hillairet & Wittwer, 2013](#); [Guillod & Wittwer, 2015](#)). The only notable exception is the solutions constructed by [Hamel \(1917, §11\)](#),

$$\bar{\mathbf{u}}_{\Phi, \mu, A} = \frac{\Phi}{2\pi r} \mathbf{e}_r + \left( \frac{\mu}{2\pi r} + A\gamma(r) \right) \mathbf{e}_\theta, \quad \gamma(r) = \begin{cases} r^{1+\frac{\Phi}{2\pi}}, & \Phi \neq -4\pi, \\ \frac{\log r}{r}, & \Phi = -4\pi. \end{cases} \quad (3)$$

where  $A, \Phi, \mu \in \mathbb{R}$ . For  $A \neq 0$  and  $-4\pi \leq \Phi < -2\pi$ , the Hamel solution (3) is supercritically decaying, otherwise it is critically decaying.

In contrast to the three-dimensional case studied by [Heywood \(1970\)](#); [Borchers & Miyakawa \(1995\)](#), in two dimensions the Hardy inequality has a logarithmic correction and therefore cannot be used to show that any critically decaying steady solution satisfies our general hypothesis. We will show that the critically decaying flux carrier  $\bar{\mathbf{u}}_{\Phi, 0, 0}$  with  $|\Phi| < 2\pi$  satisfies the general hypothesis as well as any subcritical solutions. We remark that our stability result breaks down with respect to the size of the flux exactly at the same value ( $\Phi = -2\pi$ ) where the uniqueness of the steady solution of (2) is invalidated by the Hamel solution (3), see for example [Galdi \(2011, §XII.2\)](#). Moreover, we show that our general hypothesis is satisfied if everything is assumed to be centrally symmetric. This improves the results of [Galdi & Yamazaki \(2015\)](#); [Yamazaki \(2016\)](#) which require axial symmetries with respect to both coordinates axes. However, we will show that the pure rotating solution  $\bar{\mathbf{u}}_{0, \mu, 0}$  with  $\mu \neq 0$  does not satisfy our general hypothesis without assuming the central symmetry. Very recently, [Maekawa \(2016\)](#) announced the asymptotic stability of the pure rotating solution in the exterior of a disk under the assumption that both  $\mu$  and the  $L^2$ -perturbations are small, by calculating the spectrum of the linearized operator almost explicitly. To our knowledge, this is the only result together with the present ones showing the stability of a nontrivial steady state for the two-dimensional Navier-Stokes equations.

**Notations** The space of smooth solenoidal functions having compact support in  $\Omega$  is denoted by  $C_{0, \sigma}^\infty(\Omega)$ . The completion of  $C_{0, \sigma}^\infty(\Omega)$  in the natural norm associated to  $L^2(\Omega)$ ,  $H^1(\Omega)$ , and

$\dot{H}^1(\Omega)$  are denoted by  $L_\sigma^2(\Omega)$ ,  $H_{0,\sigma}^1(\Omega)$ , and  $\dot{H}_{0,\sigma}^1(\Omega)$  respectively. The Sobolev space  $H_\sigma^1(\Omega)$  and its homogeneous counterpart  $\dot{H}_\sigma^1(\Omega)$  denote the space of weakly divergence-free vector fields respectively in  $H^1(\Omega)$  and in  $\dot{H}^1(\Omega)$ . See for example Galdi (2011, Chapters II & III) for the standard properties of these spaces.

## 2 Main results

We make the following general hypothesis which is required to show the stability by the energy method:

**Hypothesis 1.** Assume there exists  $\delta \in [0, 1)$  and a solution  $\bar{\mathbf{u}} \in \dot{H}_\sigma^1(\Omega)$  of (2) (for some  $\mathbf{u}^*$ ) such that for all  $\mathbf{v} \in \dot{H}_{0,\sigma}^1(\mathbb{R}^2)$ ,

$$(\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}}) \leq \delta \|\nabla \mathbf{v}\|_2^2.$$

*Remark 2.* The linearized operator of (2) around  $\bar{\mathbf{u}}$  is defined on  $H_{0,\sigma}^1(\Omega)$  by

$$\mathcal{L}\mathbf{v} = -\Delta \mathbf{v} + \mathbb{P}(\bar{\mathbf{u}} \cdot \nabla \mathbf{v}) + \mathbb{P}(\mathbf{v} \cdot \nabla \bar{\mathbf{u}}),$$

where  $\mathbb{P}$  is the Leray projection. This operator satisfies

$$(\mathcal{L}\mathbf{v}, \mathbf{v}) = \|\nabla \mathbf{v}\|_2^2 - (\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}}),$$

and therefore is positive definite if and only if hypothesis 1 holds.

By defining

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{v}, \quad \mathbf{u}_0 = \bar{\mathbf{u}} + \mathbf{v}_0,$$

the original system (1) becomes

$$\begin{aligned} \partial_t \mathbf{v} + \bar{\mathbf{u}} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \bar{\mathbf{u}} + \mathbf{v} \cdot \nabla \mathbf{v} &= \Delta \mathbf{v} - \nabla q, \quad \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}|_{t=0} &= \mathbf{v}_0, \quad \mathbf{v}|_{\partial\Omega} = \mathbf{0}. \end{aligned} \quad (4)$$

By using hypothesis 1, the existence of weak solutions can be shown by the standard method:

**Definition 3.** A weak solution  $\mathbf{v}$  of (4) is a vector-field  $\mathbf{v} \in L^\infty(0, \infty; L_\sigma^2(\Omega)) \cap L^2(0, \infty; \dot{H}_{0,\sigma}^1(\Omega))$  such that

$$\begin{aligned} (\mathbf{v}(t), \varphi(t)) + \int_s^t [(\nabla \mathbf{v}, \nabla \varphi) + (\bar{\mathbf{u}} \cdot \nabla \mathbf{v}, \varphi) + (\mathbf{v} \cdot \nabla \bar{\mathbf{u}}, \varphi) + (\mathbf{v} \cdot \nabla \mathbf{v}, \varphi)] d\tau \\ = (\mathbf{v}(s), \varphi(s)) + \int_s^t (\mathbf{v}, \dot{\varphi}) d\tau, \end{aligned} \quad (5)$$

for all  $s \geq t \geq 0$  and  $\varphi \in C^1(0, \infty, C_{0,\sigma}^\infty(\Omega))$ .

**Theorem 4.** Assuming hypothesis 1 holds, for  $\mathbf{v}_0 \in L_\sigma^2(\Omega)$  and  $T > 0$ , there exists a weak solution  $\mathbf{v}$  of (4) in the energy space

$$X_T = L^\infty(0, T; L_\sigma^2(\Omega)) \cap L^2(0, T; \dot{H}_{0,\sigma}^1(\Omega)),$$

satisfying the energy inequality

$$\|\mathbf{v}(t)\|_2^2 + (1 - \delta) \int_s^t \|\nabla \mathbf{v}(\tau)\|_2^2 d\tau \leq \|\mathbf{v}(s)\|_2^2. \quad (6)$$

The main result of the paper is the asymptotic  $L^2$ -stability of the steady solutions satisfying hypothesis 1.

**Theorem 5.** Assuming hypothesis 1 holds, for  $\mathbf{v}_0 \in L_\sigma^2(\Omega)$  and  $\mathbf{v}$  a weak solution of (4) on  $(0, \infty)$  satisfying (6), then

$$\lim_{t \rightarrow \infty} \|\mathbf{v}(t)\|_2 = 0.$$

### 3 Conditions for the validity of the main hypothesis

In this section, we will deduce some explicit conditions on  $\bar{\mathbf{u}}$  which ensure the validity of our main hypothesis 1 and therefore its stability. We will prove that the critically decaying flux carrier  $\bar{\mathbf{u}}_{\Phi,0,0}$  and subcritically decaying steady solutions satisfy hypothesis 1 under suitable smallness assumptions. We will prove that the steady solution  $\bar{\mathbf{u}}_{0,\mu,0}$  corresponding to a rotating cylinder doesn't satisfy hypothesis 1. Moreover under the central symmetry assumption, any small critically decaying solution satisfies hypothesis 1.

The most simple case concerning subcritically decaying steady solutions is obtained by applying the Hardy inequality:

**Lemma 6.** If for some  $\delta > 0$ ,

$$\sup_{\mathbf{x} \in \Omega} (|\mathbf{x}| \log(\varepsilon^{-1} |\mathbf{x}|) |\bar{\mathbf{u}}(\mathbf{x})|) \leq \frac{\delta}{2}, \quad (7)$$

then for all  $\mathbf{v} \in \dot{H}_{0,\sigma}^1(\Omega)$ ,

$$|(\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}})| \leq \delta \|\nabla \mathbf{v}\|_2^2.$$

*Proof.* This is a simple consequence of the Hardy inequality (Galdi, 2011, Theorem II.6.1),

$$\left\| \frac{\mathbf{v}}{|\mathbf{x}| \log(\varepsilon^{-1} |\mathbf{x}|)} \right\| \leq 2 \|\nabla \mathbf{v}\|_2.$$

□

By assuming that the field  $\mathbf{v}$  is centrally symmetric,

$$\mathbf{v}(\mathbf{x}) = -\mathbf{v}(-\mathbf{x}),$$

we can obtain an Hardy inequality without logarithmic correction, and therefore can treat the stability of critically decaying steady solutions:

**Lemma 7.** There exists some constant  $C > 0$  depending only on the domain  $\Omega$ , such that if

$$\sup_{\mathbf{x} \in \Omega} (|\mathbf{x}| |\bar{\mathbf{u}}(\mathbf{x})|) \leq \frac{\delta}{C}, \quad (8)$$

for some  $\delta > 0$ , then for any centrally symmetric  $\mathbf{v} \in \dot{H}_{0,\sigma}^1(\Omega)$ ,

$$|(\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}})| \leq \delta \|\nabla \mathbf{v}\|_2^2.$$

*Proof.* It suffices to prove the following Hardy inequality

$$\left\| \frac{\mathbf{v}}{|\mathbf{x}|} \right\|_2 \leq C \|\nabla \mathbf{v}\|_2, \quad (9)$$

for all centrally symmetric  $\mathbf{v} \in \dot{H}_{0,\sigma}^1(\Omega)$ . Let  $R > 0$  be such that  $B \subset B(\mathbf{0}, R)$ . We denote by  $B_n$  the ball  $B_n = B(\mathbf{0}, nR)$  and by  $S_n$  the shell

$$S_0 = B_1 \setminus B, \quad S_n = B_{2n} \setminus B_n, \quad \text{for } n \geq 1.$$

Since  $\mathbf{v}$  is zero on  $\partial B$ , by using the Poincaré inequality in  $S_0$ , there exists a constant  $C_0 > 0$  such that

$$\|\mathbf{u}; L^2(S_0)\|^2 \leq C_0 \|\nabla \mathbf{u}; L^2(S_0)\|^2.$$

Since  $\mathbf{v}$  is centrally symmetric, we have

$$\int_{\gamma} \mathbf{v} = 0,$$

for  $\gamma$  any centrally symmetric smooth curve. Therefore for  $n \geq 1$ ,

$$\int_{S_n} \mathbf{v} = 0,$$

and by using the Poincaré inequality in  $S_n$ , there exists  $C_n > 0$  such that

$$\|\mathbf{v}; L^2(S_n)\|^2 \leq C_n \|\nabla \mathbf{v}; L^2(S_n)\|^2.$$

By hypothesis  $|\mathbf{x}| \geq \varepsilon$  so we obtain

$$\|\mathbf{v}/|\mathbf{x}|; L^2(S_n)\|^2 \leq \frac{C_n}{\varepsilon} \|\nabla \mathbf{v}; L^2(S_n)\|^2.$$

But the domains  $S_n$  are scaled versions of  $S_1$ , *i.e.*  $S_n = nS_1$  for  $n \geq 1$  and therefore, since the two norms in the previous inequality are scale invariant, we obtain that  $C_n = C_1$ , for  $n \geq 1$ . Now we have for  $N \geq 1$ ,

$$\begin{aligned} \|\mathbf{v}/|\mathbf{x}|; L^2(B_{2N} \setminus B)\|^2 &= \sum_{n=0}^N \|\mathbf{v}/|\mathbf{x}|; L^2(S_n)\|^2 \leq \frac{1}{\varepsilon} \sum_{n=0}^N C_n \|\nabla \mathbf{u}; L^2(S_n)\|^2 \\ &\leq \frac{C_0 + C_1}{\varepsilon} \sum_{n=0}^N \|\nabla \mathbf{u}; L^2(S_n)\|^2 \leq \frac{C_0 + C_1}{\varepsilon} \|\nabla \mathbf{u}; L^2(B_{2N} \setminus B)\|^2. \end{aligned}$$

Finally, by taking the limit  $N \rightarrow \infty$ , (9) is proven where  $C^2 = \varepsilon^{-1}(C_0 + C_1)$  depends only on  $\Omega$ .  $\square$

Finally, in the next two lemmas, we investigate the validity of hypothesis 1 without symmetry assumptions for the two critically decaying harmonic functions. We prove that hypothesis 1 is verified for the flux carrier under some conditions but never for the pure rotating solution.

**Lemma 8** (Russo, 2011, Lemma 3). *For  $\Phi \in \mathbb{R}$ , if*

$$\bar{\mathbf{u}}_{\Phi} = \frac{\Phi}{2\pi} \frac{\mathbf{e}_r}{r},$$

*then for all  $\mathbf{v} \in \dot{H}_{0,\sigma}^1(\Omega)$ ,*

$$|(\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}}_{\Phi})| \leq \frac{|\Phi|}{2\pi} \|\nabla \mathbf{v}\|_2^2.$$

*Proof.* We have

$$\bar{\mathbf{u}}_{\Phi,0,0} = \frac{\Phi}{2\pi} \nabla \log r,$$

so by integrating by part, we have

$$(\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}}_{\Phi,0,0}) = \frac{\Phi}{2\pi} (\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}), \log r).$$

By using Coifman *et al.* (1993, Theorem II.1),  $\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = \nabla \mathbf{v} : (\nabla \mathbf{v})^T$  is in the Hardy space  $\mathcal{H}^1(\Omega)$ , and since  $\log r \in BMO(\Omega)$ , we obtain that  $(\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}}_{\Phi})$  is a continuous bilinear form on  $\dot{H}_{0,\sigma}^1(\Omega)$ . An explicit calculation performed by Russo (2011, Lemma 3) and reproduced in Galdi (2011, Remark X.4.2) determines the value of the constant,

$$|(\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}}_{\Phi,0,0})| \leq \frac{|\Phi|}{2\pi} \|\nabla \mathbf{v}\|_2^2.$$

$\square$

The following lemma is given as an example that not all critically decaying steady solutions satisfy hypothesis 1, but can be skipped as it will not be used later on.

**Lemma 9.** For  $\mu \neq 0$ , if

$$\bar{\mathbf{u}}_{0,\mu,0} = \frac{\mu}{2\pi} \frac{\mathbf{e}_\theta}{r},$$

then for any  $\delta > 0$ , there exists  $\mathbf{v} \in H_{0,\sigma}^1(\Omega)$  such that

$$(\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}}_{0,\mu,0}) \geq \delta \|\nabla \mathbf{v}\|_2^2.$$

*Proof.* Without loss of generality, we assume  $\mu = 2\pi$ . For  $\alpha > 0$  small, we define the following stream function and the corresponding velocity field,

$$\psi_\alpha = r^{\cos \alpha} \cos(\theta - \sin \alpha \log r), \quad \mathbf{u}_\alpha = \nabla \wedge \psi_\alpha.$$

Then an explicit calculation, shows that

$$\int_{\mathbb{R}^2 \setminus B(\mathbf{0},1)} |\nabla \mathbf{u}_\alpha|^2 = 4\pi. \quad (10)$$

On the other hand, we have

$$\mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \cdot \bar{\mathbf{u}}_{0,\mu,0} = \frac{\sin 2\alpha}{2} r^{-4+2\cos \alpha},$$

so

$$\int_{\mathbb{R}^2 \setminus B(\mathbf{0},1)} (\mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \cdot \bar{\mathbf{u}}_{0,\mu,0}) = \frac{\pi \sin 2\alpha}{2 - 2\cos \alpha}. \quad (11)$$

Since  $\|\mathbf{u}_\alpha\|_\infty$  and  $\||\mathbf{x}|^{-1} \psi_\alpha\|_\infty$  are uniformly bounded in  $\alpha$ , if  $\mathbf{u}_\alpha$  were in  $H_{0,\sigma}^1(\Omega)$ , the lemma would be proven by taking  $\alpha \rightarrow 0$ . In the following, we will make a suitable cutoff of  $\mathbf{u}_\alpha$  to make its support compact in  $\Omega$ .

Let  $\chi : \Omega \rightarrow [0, 1]$  be a smooth cutoff function such that  $\chi = 0$  on  $\Omega \cap \bar{B}(\mathbf{0}, R)$  and  $\chi = 1$  on  $\Omega \setminus B(\mathbf{0}, 2R)$ . Let  $\eta : \mathbb{R} \rightarrow [0, 1]$  be a smooth cutoff function such that  $\eta(r) = 1$  if  $r \leq 1$  and  $\eta(r) = 0$  if  $r \geq 2$ . For  $k > 0$  large enough, let  $\eta_k$  be defined by

$$\eta_k(\mathbf{x}) = \chi(\mathbf{x}) \eta\left(\frac{\log \log |\mathbf{x}|}{\log \log k}\right).$$

Therefore,  $\eta_k$  has support in  $B(\mathbf{0}, K) \setminus \bar{B}(\mathbf{0}, R)$  where  $K = e^{\log^2 k}$  and  $\eta_k(\mathbf{x}) = 1$  if  $2R \leq |\mathbf{x}| \leq k$ . Moreover the decay at infinity is the following,

$$|\nabla \eta_k(\mathbf{x})| + |\mathbf{x}| |\nabla^2 \eta_k(\mathbf{x})| \leq \frac{C_\eta}{\log \log k} \frac{1}{|\mathbf{x}| \log |\mathbf{x}|}, \quad (12)$$

where  $C_\eta > 0$  is independent of  $k$ . We now define

$$\mathbf{u}_{\alpha,k} = \nabla \wedge (\eta_k \psi_\alpha) = \eta_k \mathbf{u}_\alpha + \psi_\alpha (\nabla \wedge \eta_k),$$

so

$$\nabla \mathbf{u}_{\alpha,k} = \eta_k \nabla \mathbf{u}_\alpha + \nabla \eta_k \otimes \mathbf{u}_\alpha + \mathbf{u}_\alpha^\perp \otimes (\nabla \wedge \eta_k) + \psi_\alpha \nabla (\nabla \wedge \eta_k).$$

Since  $\|\mathbf{u}_\alpha\|_\infty$  and  $\||\mathbf{x}|^{-1} \psi_\alpha\|_\infty$  are uniformly bounded in  $\alpha$ , in view of (12) and (10), we have

$$\|\nabla \mathbf{u}_{\alpha,k}\|_2 \leq \|\eta_k\|_\infty \|\nabla \mathbf{u}_\alpha\|_2 + \|\mathbf{u}_\alpha\|_\infty \|\nabla \eta_k\|_2 + \||\mathbf{x}|^{-1} \psi_\alpha\|_\infty \||\mathbf{x}| \nabla^2 \eta_k\|_2 \leq C, \quad (13)$$

for some  $C_u > 0$  independent of  $\alpha$  and  $k$ .

To analyze the other term, we split the domain  $\Omega$  into  $\Omega_1 = \Omega \cap B(\mathbf{0}, 2R)$  and  $\Omega_2 = \mathbb{R}^2 \setminus B(\mathbf{0}, 2R)$  and define

$$I_i(\alpha, k) = \int_{\Omega_i} (\mathbf{u}_{\alpha, k} \cdot \nabla \mathbf{u}_{\alpha, k} \cdot \bar{\mathbf{u}}_{0, \mu, 0}),$$

for  $i = 1, 2$ . The functions are independent of  $k$  in  $\Omega_1$ , so there exists a constant  $C_1 > 0$  independent of  $\alpha$  and  $k$  such that

$$|I_1(\alpha, k)| \leq C_1.$$

In  $\Omega_2$ , the cutoff-function  $\eta_k$  is radially symmetric and an explicit calculation shows that

$$\begin{aligned} I_2(\alpha, k) &= 2\pi \int_{2R}^{\infty} r^{-3+2\cos\alpha} \left( \sin 2\alpha \eta_k^2 + \sin \alpha \eta_k r \nabla \eta_k \cdot \mathbf{e}_r \right) dr \\ &\geq 2\pi \sin 2\alpha \int_{2R}^k r^{-3+2\cos\alpha} dr - \frac{2\pi C_\eta \sin \alpha}{\log \log k} \left| \int_k^K r^{-3+2\cos\alpha} \frac{dr}{\log r} \right|, \end{aligned}$$

where we used (12) for the last step. We have

$$\left| \int_k^K r^{-3+2\cos\alpha} \frac{dr}{\log r} \right| \leq \left| \int_e^K \frac{dr}{r \log r} \right| \leq \log \log K \leq 2 \log \log k,$$

and moreover, by choosing

$$k_\alpha = 2^{\frac{1}{2(1-\cos\alpha)}}, \quad (14)$$

we have

$$\int_1^{k_\alpha} r^{-3+2\cos\alpha} dr = \frac{1}{4 - 4\cos\alpha}.$$

Therefore, there exists  $C_2 > 0$  independent of  $\alpha$ , such that

$$I_2(\alpha, k_\alpha) \geq \frac{\pi \sin 2\alpha}{2 - 2\cos\alpha} - C_2.$$

Defining  $\mathbf{v}_\alpha = \mathbf{u}_{\alpha, k_\alpha}$ , we have

$$(\mathbf{v}_\alpha \cdot \nabla \mathbf{v}_\alpha, \bar{\mathbf{u}}_{0, \mu, 0}) = I_1(\alpha, k_\alpha) + I_2(\alpha, k_\alpha) \geq \frac{\pi \sin 2\alpha}{2 - 2\cos\alpha} - C_1 - C_2,$$

and therefore the lemma is proven by taking the limit  $\alpha \rightarrow 0^+$  since  $\|\nabla \mathbf{v}_\alpha\|_2 \leq C$  by (13).  $\square$

All the previous lemmas, lead to the following corollaries of theorem 5:

**Corollary 10.** *Let assume that  $\bar{\mathbf{u}} = \bar{\mathbf{u}}_\Phi + \bar{\mathbf{u}}_\delta \in \dot{H}_\sigma^1(\Omega)$ , where  $\bar{\mathbf{u}}_\delta$  satisfies (7) for some  $\delta > 0$ , is a steady solution of (2). If  $|\Phi| + 2\pi\delta < 2\pi$ , then  $\bar{\mathbf{u}}$  is asymptotically stable with respect to any  $L^2$ -perturbation.*

*Proof.* By lemmas 6 and 8, hypothesis 1 is satisfied, so theorem 5 holds.  $\square$

**Corollary 11.** *Let assume that the domain  $\Omega$  is invariant under the central symmetry  $\mathbf{x} \mapsto -\mathbf{x}$ , and that  $\bar{\mathbf{u}} \in \dot{H}_\sigma^1(\Omega)$  is a centrally symmetric steady solution of (2) satisfying (8) for some  $\delta > 0$ . If  $\delta < 1$ , then  $\bar{\mathbf{u}}$  is asymptotically stable with respect to any centrally symmetric  $L^2$ -perturbation.*

*Proof.* The Navier-Stokes equations (1) and (2) are invariant under the central symmetry and therefore, all the spaces can be restricted to centrally symmetric solutions. Then the result follows by applying lemma 7 and theorem 5.  $\square$

## 4 Existence of a weak solution

The proof of the existence of weak solutions follows the standard method for showing the existence of weak solutions, see for example [Temam \(1977, Theorem 3.1\)](#), so only the main steps and sketched below.

*Proof of theorem 4.* Let  $\{\varphi_i\}_{i \geq 1} \subset C_{0,\sigma}^\infty(\Omega)$  be a sequence which is dense and orthonormal in  $H_{0,\sigma}^1(\Omega)$ . For each  $n$ , we define an approximate solution  $\mathbf{v}_n$  with initial data,

$$\mathbf{v}_n = \sum_{i=1}^n \xi_{in} \varphi_i, \quad \mathbf{v}_n(0) = \sum_{i=1}^n (\mathbf{v}_0, \varphi_i) \varphi_i,$$

which satisfies

$$(\partial_t \mathbf{v}_n, \varphi_i) + (\nabla \mathbf{v}_n, \nabla \varphi_i) + (\bar{\mathbf{u}} \cdot \nabla \mathbf{v}_n, \varphi_i) + (\mathbf{v}_n \cdot \nabla \bar{\mathbf{u}}, \varphi_i) + (\mathbf{v}_n \cdot \nabla \mathbf{v}_n, \varphi_i) = 0, \quad (15)$$

for all  $i \in \{0, \dots, n\}$ . We now obtain the a priori estimates. By multiplying (15) by  $\xi_{in}$  and summing over  $i$ ,

$$(\partial_t \mathbf{v}_n, \mathbf{v}_n) + (\nabla \mathbf{v}_n, \nabla \mathbf{v}_n) + (\mathbf{v}_n \cdot \nabla \bar{\mathbf{u}}, \mathbf{v}_n) = 0.$$

Therefore by using hypothesis 1, we have

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}_n\|_2^2 + (1 - \delta) \|\nabla \mathbf{v}_n\|_2^2 \leq 0. \quad (16)$$

Let  $i \geq 1$  and  $\Omega_i$  denotes the support of  $\varphi_i$ . Therefore the sequence  $\{\mathbf{v}_n\}_{n \geq 1}$  is bounded in  $X_T$ , so there exists a sub-sequence, also denoted by  $\{\mathbf{v}_n\}_{n \geq 1}$ , which converges to  $\mathbf{v} \in X_T$  weakly in  $L^2(0, T; \dot{H}_{0,\sigma}^1(\Omega_i))$  and strongly in  $L^2(0, T; L^2(\Omega_i))$ . Multiplying (15) by  $\psi \in C^1([0, T])$ , we obtain

$$- (\mathbf{v}_n, \dot{\psi} \varphi_i) + (\nabla \mathbf{v}_n, \nabla \psi \varphi_i) + (\bar{\mathbf{u}} \cdot \nabla \mathbf{v}_n, \psi \varphi_i) + (\mathbf{v}_n \cdot \nabla \bar{\mathbf{u}}, \psi \varphi_i) + (\mathbf{v}_n \cdot \nabla \mathbf{v}_n, \psi \varphi_i) = 0.$$

By integrating by parts, the third and the last term, we can pass to the limit and obtain the existence of a weak solution  $\mathbf{v}$  in  $X_T$ . By integrating the energy inequality for the approximate solution (16) and passing to the limit, we obtain (6).  $\square$

## 5 Linearized system

In this section, we study the linear system

$$\begin{aligned} \partial_t \mathbf{v} + \bar{\mathbf{u}} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \bar{\mathbf{u}} &= \Delta \mathbf{v} - \nabla q, & \nabla \cdot \mathbf{v} &= 0, \\ \mathbf{v}|_{t=0} &= \mathbf{v}_0, & \mathbf{v}|_{\partial\Omega} &= \mathbf{0}. \end{aligned} \quad (17)$$

Throughout this section we always assume that hypothesis 1 holds. If  $\mathbb{P}$  denotes the Leray projection, we define the following operator

$$\mathcal{L}\mathbf{v} = -\Delta \mathbf{v} + \mathbb{P}(\bar{\mathbf{u}} \cdot \nabla \mathbf{v}) + \mathbb{P}(\mathbf{v} \cdot \nabla \bar{\mathbf{u}}),$$

and its adjoint on  $L_\sigma^2(\Omega)$ ,

$$\mathcal{L}^* \mathbf{v} = -\Delta \mathbf{v} - \bar{\mathbf{u}} \cdot \nabla \mathbf{v} + (\nabla \bar{\mathbf{u}})^T \mathbf{v}.$$

More precisely they are defined by the following bilinear forms on  $H_{0,\sigma}^1(\Omega)$ ,

$$\begin{aligned} a_{\mathcal{L}}(\mathbf{v}, \varphi) &= (\nabla \mathbf{v}, \nabla \varphi) + (\bar{\mathbf{u}} \cdot \nabla \mathbf{v}, \varphi) + (\mathbf{v} \cdot \nabla \bar{\mathbf{u}}, \varphi), \\ a_{\mathcal{L}^*}(\mathbf{v}, \varphi) &= (\nabla \mathbf{v}, \nabla \varphi) - (\bar{\mathbf{u}} \cdot \nabla \mathbf{v}, \varphi) + (\varphi \cdot \nabla \bar{\mathbf{u}}, \mathbf{v}). \end{aligned}$$

We show that  $-\mathcal{L}$  and  $-\mathcal{L}^*$  generate strongly continuous semigroups on  $L_\sigma^2(\Omega)$  and some standard corollaries:



**Proposition 12.** *The closure in  $L^2_\sigma(\Omega)$  of the operators  $\mathcal{L}$  and  $\mathcal{L}^*$  are infinitesimal generators of analytic semigroups on  $L^2_\sigma(\Omega)$ .*

*Proof.* First of all, the forms  $a_{\mathcal{L}}$  and  $a_{\mathcal{L}^*}$  are bounded on  $H^1_{0,\sigma}(\Omega)$ ,

$$\begin{aligned} |a_{\mathcal{L}}(\mathbf{v}, \boldsymbol{\varphi})| &\leq \|\nabla \mathbf{v}\|_2 \|\nabla \boldsymbol{\varphi}\|_2 + \|\bar{\mathbf{u}}\|_\infty \|\nabla \mathbf{v}\|_2 \|\boldsymbol{\varphi}\|_2 + \|\nabla \bar{\mathbf{u}}\|_2 \|\mathbf{v}\|_4 \|\boldsymbol{\varphi}\|_4, \\ |a_{\mathcal{L}^*}(\mathbf{v}, \boldsymbol{\varphi})| &\leq \|\nabla \mathbf{v}\|_2 \|\nabla \boldsymbol{\varphi}\|_2 + \|\bar{\mathbf{u}}\|_\infty \|\nabla \mathbf{v}\|_2 \|\boldsymbol{\varphi}\|_2 + \|\nabla \bar{\mathbf{u}}\|_2 \|\mathbf{v}\|_4 \|\boldsymbol{\varphi}\|_4, \end{aligned}$$

since  $H^1_{0,\sigma}(\Omega)$  is continuously embedded in  $L^4(\Omega)$ . Moreover by using hypothesis 1,

$$a_{\mathcal{L}}(\mathbf{v}, \mathbf{v}) = a_{\mathcal{L}^*}(\mathbf{v}, \mathbf{v}) = \|\nabla \mathbf{v}\|_2^2 - (\mathbf{v} \cdot \nabla \mathbf{v}, \bar{\mathbf{u}}) \geq (1 - \delta) \|\nabla \mathbf{v}\|_2^2. \quad (18)$$

Therefore, by using for example Karch & Pilarczyk (2011, Proposition 4.1) or the references therein, we obtain that  $-\mathcal{L}$  and  $-\mathcal{L}^*$  generate analytic semigroups on  $L^2_\sigma(\Omega)$ .  $\square$

We have the following standard corollaries:

**Corollary 13.** *For any  $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ , we have*

$$\|e^{-t\mathcal{L}}\mathbf{v}_0\|_2 \leq \|\mathbf{v}_0\|_2, \quad \lim_{t \rightarrow \infty} \|e^{-t\mathcal{L}}\mathbf{v}_0\|_2 = 0, \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|e^{-s\mathcal{L}}\mathbf{v}_0\|_2 ds = 0.$$

*Proof.* The solution  $\mathbf{v}(t) = e^{-t\mathcal{L}}\mathbf{v}_0$  satisfies the following energy inequality,

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}\|_2^2 + (1 - \delta) \|\nabla \mathbf{v}\|_2^2 = 0.$$

The first bound is proven by integrating from 0 to  $t$ .

For the second bound, since the range of  $\mathcal{L}$  is dense in  $L^2_\sigma(\Omega)$ , for every  $\varepsilon > 0$  there exists  $\boldsymbol{\varphi} \in \mathcal{D}(\mathcal{L})$  such that  $\|\mathbf{v}_0 - \mathcal{L}\boldsymbol{\varphi}\|_2 \leq \varepsilon$ . Therefore, by using the first bound,

$$\|e^{-t\mathcal{L}}\mathbf{v}_0\|_2 \leq \|e^{-t\mathcal{L}}(\mathbf{v}_0 - \mathcal{L}\boldsymbol{\varphi})\|_2 + \|\mathcal{L}e^{-t\mathcal{L}}\boldsymbol{\varphi}\|_2 \leq \varepsilon + Ct^{-1} \|\boldsymbol{\varphi}\|,$$

so the second bound is proven.

Substituting  $s = \tau t$ , we have

$$\frac{1}{t} \int_0^t \|e^{-s\mathcal{L}}\mathbf{v}_0\|_2 ds = \int_0^1 \|e^{-\tau\mathcal{L}}\mathbf{v}_0\|_2 d\tau \leq \|\mathbf{v}_0\|_2,$$

since the semigroup is contracting. Therefore, the third bound follows by using the second bound together with the dominated convergence theorem.  $\square$

**Corollary 14.** *For any  $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ , we have*

$$\|\nabla e^{-t\mathcal{L}^*}\mathbf{v}_0\|_2 \leq Ct^{-1/2} \|\mathbf{v}_0\|_2.$$

*Proof.* Using (18), for  $\mathbf{v} \in L^2_\sigma(\Omega)$ , we have

$$(1 - \delta) \|\nabla \mathbf{v}\|_2^2 \leq a_{\mathcal{L}^*}(\mathbf{v}, \mathbf{v}) = (\mathcal{L}^*\mathbf{v}, \mathbf{v}) \leq \|\mathcal{L}^*\mathbf{v}\|_2 \|\mathbf{v}\|_2.$$

Therefore, taking  $\mathbf{v} = e^{-t\mathcal{L}^*}\mathbf{v}_0$ , we obtain

$$\|\nabla \mathbf{v}\|_2^2 \leq (1 - \delta)^{-1} \|\mathcal{L}^* e^{-t\mathcal{L}^*}\mathbf{v}_0\|_2 \|\mathbf{v}_0\|_2 \leq C^2 t^{-1} \|\mathbf{v}_0\|_2^2.$$

$\square$

## 6 Asymptotic stability

Using the properties of the linearized system, we now prove the result on the asymptotic stability (theorem 5).

**Lemma 15.** There exists  $C > 0$  such that for all  $\mathbf{v} \in H_{0,\sigma}^1(\Omega)$  and  $\varphi \in L_\sigma^2(\Omega)$ ,

$$\left( \mathbf{v} \cdot \nabla \mathbf{v}, e^{-t\mathcal{L}^*} \varphi \right) \leq Ct^{-1/2} \|\mathbf{v}\|_2 \|\nabla \mathbf{v}\|_2 \|\varphi\|_2 .$$

*Proof.* Using corollary 14, we have

$$\left| \left( \mathbf{v} \cdot \nabla \mathbf{v}, e^{-t\mathcal{L}^*} \varphi \right) \right| = \left| \left( \mathbf{v} \cdot \nabla e^{-t\mathcal{L}^*} \varphi, \mathbf{v} \right) \right| \leq \|\mathbf{v}\|_4^2 \|\nabla e^{-t\mathcal{L}^*} \varphi\|_2 \leq Ct^{-1/2} \|\mathbf{v}\|_4^2 \|\varphi\|_2 .$$

The result follows by applying the Sobolev inequality  $\|\mathbf{v}\|_4^4 \leq 2 \|\mathbf{v}\|_2^2 \|\nabla \mathbf{v}\|_2^2$ .  $\square$

The following lemma will be crucial in the proof of the asymptotic stability without using any spectral decomposition.

**Lemma 16.** For a nonnegative function  $f \in L^2(0, \infty)$ , we have

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \int_0^s (s - \tau)^{-1/2} f(\tau) d\tau ds = 0 .$$

*Proof.* We first prove that

$$\lim_{t \rightarrow \infty} \|\chi_t f\|_1 = 0 , \quad \text{where} \quad \chi_t = \frac{1}{\sqrt{t}} \mathbb{1}_{[0,t]} .$$

Since  $\|\chi_t\|_2 = 1$  and  $\chi_t$  converges pointwise to 0 as  $t \rightarrow \infty$ , we obtain that  $\chi_t$  converges weakly to 0 in  $L^2(0, \infty)$  as  $t \rightarrow \infty$ , see [Hewitt & Stromberg \(1965, Theorem 13.44\)](#). Therefore  $\lim_{t \rightarrow \infty} \|\chi_t f\|_1 = 0$  since  $f \in L^2(0, \infty)$ .

By interchanging the order of integrations, we have

$$\begin{aligned} I(t) &= \frac{1}{t} \int_0^t \int_0^s (s - \tau)^{-1/2} f(\tau) d\tau ds = \frac{1}{t} \int_0^t \int_\tau^t (s - \tau)^{-1/2} f(\tau) ds d\tau \\ &= \frac{2}{t} \int_0^t (t - \tau)^{1/2} f(\tau) d\tau \leq \frac{2}{\sqrt{t}} \int_0^t f(\tau) d\tau = 2 \|\chi_t f\|_1 , \end{aligned}$$

where the Hölder inequality justifies the interchange of the integrations, since  $\|\chi_t f\|_1 \leq \|\chi_t\|_2 \|f\|_2 \leq \|f\|_2$ . The lemma is proven since  $\lim_{t \rightarrow \infty} I(t) \leq 2 \lim_{t \rightarrow \infty} \|\chi_t f\|_1 = 0$ .  $\square$

*Proof of theorem 5.* Let  $\varphi \in C_{0,\sigma}^\infty(\Omega)$ ,  $s > 0$ , and

$$\psi(\tau) = e^{-(s-\tau)\mathcal{L}^*} \varphi ,$$

for  $\tau \in (0, s)$ . Since

$$\begin{aligned} \int_0^s (\mathbf{v}, \dot{\psi}) d\tau &= \int_0^s (\mathbf{v}, \mathcal{L}^* \psi) d\tau = \int_0^s (\mathcal{L} \mathbf{v}, \psi) d\tau \\ &= \int_0^s \left[ (\nabla \mathbf{v}, \nabla \psi) + (\bar{\mathbf{u}} \cdot \nabla \mathbf{v}, \psi) + (\mathbf{v} \cdot \nabla \bar{\mathbf{u}}, \psi) \right] d\tau , \end{aligned}$$

by definition 3, we obtain

$$(\mathbf{v}(s), \varphi) + \int_0^s (\mathbf{v} \cdot \nabla \mathbf{v}, \psi) d\tau = (\mathbf{v}_0, \psi(0)) = (e^{-s\mathcal{L}} \mathbf{v}_0, \varphi) .$$

Taking  $\varphi = \mathbf{v}(s)$  in this expression,

$$\|\mathbf{v}(s)\|_2^2 = (\mathbf{e}^{-s\mathcal{L}}\mathbf{v}_0, \mathbf{v}(s)) - \int_0^s \left( (\mathbf{v} \cdot \nabla \mathbf{v})(\tau), \mathbf{e}^{-(s-\tau)\mathcal{L}^*} \mathbf{v}(s) \right) d\tau,$$

so by using lemma 15, we have

$$\|\mathbf{v}(s)\|_2 \leq \|\mathbf{e}^{-s\mathcal{L}}\mathbf{v}_0\|_2 + C \int_0^s (s-\tau)^{-1/2} \|\mathbf{v}(\tau)\|_2 \|\nabla \mathbf{v}(\tau)\|_2 d\tau.$$

By the energy inequality (6), we have  $\|\mathbf{v}(\tau)\|_2 \leq \|\mathbf{v}_0\|_2$ , so

$$\|\mathbf{v}(s)\|_2 \leq \|\mathbf{e}^{-s\mathcal{L}}\mathbf{v}_0\|_2 + C \|\mathbf{v}_0\|_2 \int_0^s (s-\tau)^{-1/2} \|\nabla \mathbf{v}(\tau)\|_2 d\tau.$$

Integrating on  $s$  from 0 to  $t$  and multiplying by  $t^{-1}$ , we obtain

$$\frac{1}{t} \int_0^t \|\mathbf{v}(s)\|_2 ds \leq \frac{1}{t} \int_0^t \|\mathbf{e}^{-s\mathcal{L}}\mathbf{v}_0\|_2 ds + C \|\mathbf{v}_0\|_2 \int_0^t \int_0^s (s-\tau)^{-1/2} \|\nabla \mathbf{v}(\tau)\|_2 d\tau ds.$$

By the energy inequality,  $\|\nabla \mathbf{v}(\cdot)\|_2 \in L^2(0, \infty)$ , so in view of corollary 13 and lemma 16 we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|\mathbf{v}(s)\|_2 ds = 0.$$

Finally, since  $\|\mathbf{v}(t)\|_2$  is a non-increasing function of  $t \geq 0$ , we obtain for  $t > 0$ ,

$$\|\mathbf{v}(t)\|_2 \leq \frac{1}{t} \|\mathbf{v}(t)\|_2 \int_0^t ds \leq \frac{1}{t} \int_0^t \|\mathbf{v}(s)\|_2 ds,$$

so the proof is complete. □

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